Existence and uniqueness of a weak solution to a non-automomonous time-fractional diffusion equation (of distributed order)

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Abstract

In this contribution, we investigate an initial-boundary value problem for a fractional diffusion equation of distributed order where the coefficients of the elliptic operator are dependent on spatial and time variables. We consider a homogeneous Dirichlet boundary condition. Using a classical variational approach, we establish the existence of a unique weak solution to the problem in $\mathrm{C}\left([0,T], \mathrm{H}^1_0(\Omega)^*\right) \cap \mathrm{L}^\infty\left((0,T), \mathrm{H}^1_0(\Omega)\right)$ if the initial data belongs to $\mathrm{H}^1_0(\Omega)$. The same result is also valid for the fractional diffusion equation with Caputo derivative.

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1. Introduction

Let $\Omega \subset \mathbb{R}^d$ ($d \in \mathbb{N}$) be a bounded domain with a Lipschitz continuous boundary $\partial\Omega$. The final time is denoted by T, $Q_T := \Omega \times (0,T]$ and $\Sigma_T := \partial\Omega \times (0,T]$. Consider a general second-order linear differential operator given by

$$L(\mathbf{x},t)u(\mathbf{x},t) = -\nabla \cdot (\boldsymbol{A}(\mathbf{x},t)\nabla u(\mathbf{x},t) + \boldsymbol{b}(\mathbf{x},t)u(\mathbf{x},t)) + c(\mathbf{x},t)u(\mathbf{x},t), \tag{1}$$

where $((\mathbf{x},t) \in Q_T)$

$$\boldsymbol{A}(\mathbf{x},t) = \left(a_{i,j}(\mathbf{x},t)\right)_{i,j=1,\dots,d}, \quad \boldsymbol{b}(\mathbf{x},t) = (b_1(\mathbf{x},t), b_2(\mathbf{x},t), \dots, b_d(\mathbf{x},t)).$$

The goal of this contribution is to show the existence and uniqueness of u for given f and u_0 such that

$$\begin{cases} \left(\mathcal{D}_t^{(\mu)}u\right)(\mathbf{x},t) + (Lu)(\mathbf{x},t) &= f(\mathbf{x},t) \qquad (\mathbf{x},t) \in Q_T, \\ u(\mathbf{x},t) &= 0 \qquad (\mathbf{x},t) \in \Sigma_T, \\ u(\mathbf{x},0) &= u_0(\mathbf{x}) \qquad \mathbf{x} \in \Omega. \end{cases} \tag{2}$$

Here, $\mathcal{D}_t^{(\mu)}u$ denotes the distributed order fractional derivative defined by

$$\left(\mathcal{D}_t^{(\mu)}u\right)(\mathbf{x},t) = \int_0^1 \left(\partial_t^\beta u\right)(\mathbf{x},t)\mu(\beta)\,\mathrm{d}\beta, \qquad (\mathbf{x},t) \in Q_T, \tag{3}$$

with weight function $\mu : [0,1] \to \mathbb{R}$ satisfying

$$\mu \in \mathrm{L}^1(0,1), \quad \mu \geqslant 0, \quad \mu \not\equiv 0,$$

and with $\partial_t^\beta u$ the Caputo derivative of order $\beta \in (0,1)$ defined by

$$\left(\partial_t^\beta u\right)(\mathbf{x},t) = \int_0^t \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)}\partial_s u(\mathbf{x},s)\ \mathrm{d}s, \qquad (\mathbf{x},t) \in Q_T,$$

where Γ denotes the Gamma function. The equation in (2) is used to model ultra slow diffusion processes with a logarithmic growth of the mean square displacement, see e.g. [1, 2, 3, 4].

We only mention the most relevant results available in the literature concerning the subject of this paper. First, we discuss the papers [5, 6, 7, 8, 9]. In these papers, the considered operator L is time-independent (autonomous) and symmetric (i.e. $\boldsymbol{A}^{\top} = \boldsymbol{A}$ and $\boldsymbol{b} = \boldsymbol{0}$). In [5], an appropriate maximum principle (and as a consequence also uniqueness of a solution) for problem (2) is proved. In [6], a strong solution to problem (2) is provided when $\mu \in \mathrm{C}^1$ and $f = 0$. The asymptotic behavior of the solutions to initial-boundary-value problems for distributed order time-fractional diffusion equations (without source) is studied in [7]. The authors of [8] used the method of the eigenfunctions expansion in combination with the Laplace transform to prove the uniqueness and existence of a solution to (2) (and its analyticity in time) when $f = u_0 = 0$. In [9], the authors investigated the existence and uniqueness of a weak solution to problem (2). They characterize the weak solution to (2) as the original of the solution to the Laplace transform (of tempered distributions) of (2) with respect to the time variable.

Finally, we mention also the paper [10]. The authors showed the existence of a weak solution to (2) by an approach based on Galerkin method and energy estimates obtained for a special approximating sequence (the approximate solution is separated in the space and time variable). However, their approach (based on Banach fixed-point theorem with the solution space depending on μ) does not seem to be suitable for performing computations and the obtained continuity of the solution in the time variable is under the additional assumption that $\int_{1/2}^{1} \mu(\beta)\,\mathrm{d}\beta > 0$, see [10, Theorem 2 and 4]. Note also that in the papers above, the domain has either a smooth or $\mathrm{C}^{1,1}$ boundary, whilst in this paper a Lipschitz domain is considered.

The approach followed in this contribution illustrates that the standard procedure (time-discretization) for showing the existence of a weak solution to classical parabolic problems also can be applied on problem (2), which is a new result and also includes a scheme for computations. The weak formulation of the problem and assumptions on the data are discussed in Section 2 and the existence of a unique solution is shown in Section 3.

2. Weak formulation

We can rewrite the distributed order fractional derivative defined in Eq. (3) as follows

$$\left(\mathcal{D}_t^{(\mu)} u\right)(\mathbf{x}, t) = (k * \partial_t u)(\mathbf{x}, t), \qquad (\mathbf{x}, t) \in Q_T, \qquad \text{where} \qquad k(t) = \int_0^1 \frac{t^{-\beta}}{\Gamma(1-\beta)} \mu(\beta)\,\mathrm{d}\beta.$$

Note that the kernel k satisfies $k(t) \geqslant 0$ for $t > 0$ and it is singular at $t = 0$. The convolution kernel k belongs to $\mathrm{L}^1(0, T)$ and it is a decreasing function in time. Now, we state the assumptions on the data functions. The matrix $\boldsymbol{A} = \left(a_{ij}(\mathbf{x}, t)\right)$ is a $d \times d$ matrix-valued function such that $\boldsymbol{A} \in \left(\mathrm{L}^\infty\left(\overline{Q_T}\right)\right)^{d \times d}$ is uniformly elliptic, i.e. there exists a constant $\alpha > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(\mathbf{x}, t) \xi_i \xi_j \geqslant \alpha \left|\boldsymbol{\xi}\right|^2, \qquad \text{for a.a. } (\mathbf{x}, t) \in \overline{Q_T} \text{ and for all } \boldsymbol{\xi} \in \mathbb{R}^d.$$

Moreover, we suppose that $c \in \mathrm{L}^\infty\left(\overline{Q_T}\right)$. For the vector function $\boldsymbol{b}$, we assume that

$$\boldsymbol{b} \in \mathbf{L}^\infty\left(\overline{Q_T}\right) \qquad \text{s.t.} \qquad c(\mathbf{x}, t) - \frac{\|\boldsymbol{b}\|^2_{\mathbf{L}^\infty\left(\overline{Q_T}\right)}}{2\alpha} \geqslant 0, \qquad (\mathbf{x}, t) \in \overline{Q_T}.$$

We associate a bilinear form $\mathcal{L}$ with the differential operator L defined in (1) as follows

$$\mathcal{L}(t)\left(u(t), \varphi\right) := (Lu, \varphi) = \left(\boldsymbol{A}(t)\nabla u(t) + \boldsymbol{b}(t) u(t), \nabla\varphi\right) + \left(c(t) u(t), \varphi\right),$$

with $u(t), \varphi \in \mathrm{H}_0^1(\Omega)$. Using the properties above, we obtain that

$$\mathcal{L}(t)(u, \varphi) \leqslant C \left\|u\right\|_{\mathrm{H}_0^1(\Omega)} \left\|\varphi\right\|_{\mathrm{H}_0^1(\Omega)}, \quad \forall u, \varphi \in \mathrm{H}_0^1(\Omega), \qquad \text{and} \qquad \mathcal{L}(t)(\varphi, \varphi) \geqslant \frac{\alpha}{2} \left\|\nabla\varphi\right\|^2, \quad \forall \varphi \in \mathrm{H}_0^1(\Omega), \tag{4}$$

i.e. the bilinear form $\mathcal{L}$ is continuous and $\mathrm{H}_0^1(\Omega)$-elliptic due to the Friedrichs inequality. Next, the variational formulation of problem (2) can be defined as follows: search $u \in \mathrm{L}^2\left((0,T),\mathrm{H}_0^1(\Omega)\right)$ with $k * \partial_t u \in \mathrm{L}^2\left((0,T),\mathrm{H}_0^1(\Omega)^*\right)$ such that for a.a. $t \in (0,T)$ it holds that

$$\langle (k * \partial_t u)(t), \varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times \mathrm{H}_0^1(\Omega)} + \mathcal{L}(t)(u(t),\varphi) = \langle f(t),\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times \mathrm{H}_0^1(\Omega)}, \qquad \forall \varphi \in \mathrm{H}_0^1(\Omega). \tag{5}$$

3. Existence of a solution

The existence of a solution is shown by the aid of Rothe's method, which is a time-discretization method. The time interval $[0,T]$ is divided into $n \in \mathbb{N}$ equidistant subintervals $[t_{i-1},t_i]$ with length $\tau = \frac{T}{n} < 1$. The approximation of a function z at time $t = t_i$, $0 \leqslant i \leqslant n$, is denoted by z_i. The same notation is also used for a given function. Moreover, we approximate $\partial_t z(t_i)$, $1 \leqslant i \leqslant n$, by the backward Euler difference $\delta z_i = \frac{z_i - z_{i-1}}{\tau}$. Finally, the time discrete convolution is defined as follows

$$(k * z)(t_i) \approx (k * z)_i := \sum_{l=1}^{i} k_{i+1-l} z_l \tau.$$

Note that $(k * z)_0 := 0$. From [11, Lemma 3.2], using the evolution triple $\mathrm{H}_0^1(\Omega) \subset \mathrm{L}^2(\Omega) \cong (\mathrm{L}^2(\Omega))^* \subset \mathrm{H}_0^1(\Omega)^*$, it follows that for a sequence $(v_i)_{i\in\mathbb{N}}$ in $\mathrm{H}_0^1(\Omega)$ it holds that

$$\sum_{i=1}^{j} \langle \delta(k * v)_i, v_i\rangle_{\mathrm{H}_0^1(\Omega)^*\times \mathrm{H}_0^1(\Omega)} \tau = \sum_{i=1}^{j} (\delta(k * v)_i, v_i)\,\tau \geqslant \frac{1}{2}\left(k * \|v\|^2\right)_j + \frac{1}{2}\sum_{i=1}^{j} k_i \|v_i\|^2 \tau, \quad j \in \mathbb{N}, \tag{6}$$

with

$$\left(k * \|v\|^2\right)_j := \sum_{l=1}^{j} k_{j+1-l} \|v_l\|^2 \tau.$$

where $(\cdot,\cdot)$ denotes the standard inner product in $\mathrm{L}^2(\Omega)$ and $\|\cdot\|$ its induced norm. We approximate problem (5) at time $t = t_i$ by: Find $u_i \in \mathrm{H}_0^1(\Omega)$, $i = 1,2,\ldots,n$, such that

$$\langle (k * \delta u)_i, \varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times \mathrm{H}_0^1(\Omega)} + \mathcal{L}_i(u_i,\varphi) = \langle f_i,\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times \mathrm{H}_0^1(\Omega)}, \quad \forall \varphi \in \mathrm{H}_0^1(\Omega). \tag{7}$$

Using the time discrete convolution (3), the discrete problem can be equivalently written as

$$a_i(u_i,\varphi) := k(\tau)(u_i,\varphi) + \mathcal{L}_i(u_i,\varphi) = \langle f_i,\varphi\rangle + k(\tau)(u_{i-1},\varphi) - \sum_{l=1}^{i-1} k_{i+1-l}(u_l - u_{l-1},\varphi) =: \langle F_i,\varphi\rangle, \quad \forall \varphi \in \mathrm{H}_0^1(\Omega). \tag{8}$$

The existence of a unique solution on every time step follows from the Lax-Milgram lemma.

Lemma 3.1. *Suppose that $u_0 \in \mathrm{L}^2(\Omega)$ and $f \in \mathrm{L}^2\left((0,T),\mathrm{H}_0^1(\Omega)^*\right)$. Then, for any $i = 1,2,\ldots,n$, there exists a unique $u_i \in \mathrm{H}_0^1(\Omega)$ solving* (7).

The following two lemmas are required to ensure the existence of a solution to (5) and to prove the convergence of approximations towards this solution.

Lemma 3.2. *Let the assumptions of Lemma 3.1 be fulfilled. Then, there exist positive constants C such that for every $j = 1,2,\ldots,n$, the following relations hold*

$$\left(k * \|u\|^2\right)_j + \sum_{i=1}^{j} k_i \|u_i\|^2 \tau + \sum_{i=1}^{j} \|\nabla u_i\|^2 \tau \leqslant C \quad \text{and} \quad \sum_{i=1}^{j} \|(k * \delta u)_i\|^2_{\mathrm{H}_0^1(\Omega)^*} \tau \leqslant C.$$

Proof. We set $\varphi = u_i\tau$ in (7) and sum the result up for $i = 1, \dots, j$ with $1 \leqslant j \leqslant n$. Using the relation

$$\delta(k * u)_i = k_i u_0 + (k * \delta u)_i, \tag{9}$$

we obtain that

$$\sum_{i=1}^{j} (\delta(k * u)_i, u_i)\,\tau + \sum_{i=1}^{j} \mathcal{L}_i(u_i, u_i)\tau = \sum_{i=1}^{j} \langle f_i, u_i\rangle\tau + \sum_{i=1}^{j} k_i\,(u_0, u_i)\,\tau.$$

Employing Eq. (4) and Eq. (6) gives the first estimate. The second estimate follows from

$$\|(k * \delta u)_i\|_{\mathrm{H}_0^1(\Omega)^*} = \sup_{\|\varphi\|_{\mathrm{H}_0^1(\Omega)}=1} \left|\langle (k * \delta u)_i, \varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times \mathrm{H}_0^1(\Omega)}\right| = \sup_{\|\varphi\|_{\mathrm{H}_0^1(\Omega)}=1} |\langle f_i, \varphi\rangle - \mathcal{L}_i(u_i, \varphi)| \leqslant \|f_i\|_{\mathrm{H}_0^1(\Omega)^*} + C\,\|\nabla u_i\|\,,$$

and the result of the first estimate. $\square$

The L^∞-bound obtained in the next two lemmas is necessary to obtain the continuity in time of the solution. Here, we assume that $\boldsymbol{b} = \boldsymbol{0}$ and we consider two different condition on $\partial_t f$.

Lemma 3.3. *Let the assumptions of Lemma 3.1 be fulfilled. Moreover, assume that* $u_0 \in \mathrm{H}_0^1(\Omega)$, $\partial_t f \in \mathrm{L}^2\left((0,T), \mathrm{H}_0^1(\Omega)^*\right)$, $\boldsymbol{b} = \boldsymbol{0}$, $\partial_t \boldsymbol{A} \in \left(\mathrm{L}^\infty\left(\overline{Q_T}\right)\right)^{d\times d}$ *and* $\partial_t c \in \mathrm{L}^\infty\left(\overline{Q_T}\right)$. *Then, there exists a positive constant C such that for every $j = 1, 2, \dots, n$, the following relation holds*

$$\left\|\nabla u_j\right\|^2 + \sum_{i=1}^{j} \|\nabla u_i - \nabla u_{i-1}\|^2 \leqslant C.$$

Proof. We set $\varphi = \delta u_i\tau$ in (7) and sum the result up for $i = 1, \dots, j$ with $1 \leqslant j \leqslant n$. We obtain that

$$\sum_{i=1}^{j} \langle (k * \delta u)_i, \delta u_i\rangle_{\mathrm{H}_0^1(\Omega)^*\times \mathrm{H}_0^1(\Omega)}\tau + \sum_{i=1}^{j} \mathcal{L}_i(u_i, \delta u_i)\tau = \sum_{i=1}^{j} \langle f_i, \delta u_i\rangle_{\mathrm{H}_0^1(\Omega)^*\times \mathrm{H}_0^1(\Omega)}\tau.$$

The positivity of the first term on the left-hand side follows from [12, Eq. 3.2]. The term on the right-hand side can be estimated by using the partial summation rule $\sum_{i=1}^{j}\langle f_i, \delta u_i\rangle\tau = \langle f_j, u_j\rangle - \langle f_0, u_0\rangle - \sum_{i=1}^{j}\langle \delta f_i, u_{i-1}\rangle\tau$ and Lemma 3.2. $\square$

Lemma 3.4. *Let the assumptions of Lemma 3.1 be fulfilled. Moreover, assume that* $u_0 \in \mathrm{H}_0^1(\Omega)$, $\|\partial_t f(t)\|_{\mathrm{H}_0^1(\Omega)^*} \leqslant Ct^{-\alpha}$ *for all* $t \in (0,T]$ *and some constant* $\alpha \in (0,1)$, $\boldsymbol{b} = \boldsymbol{0}$, $\partial_t \boldsymbol{A} \in \left(\mathrm{L}^\infty\left(\overline{Q_T}\right)\right)^{d\times d}$ *and* $\partial_t c \in \mathrm{L}^\infty\left(\overline{Q_T}\right)$. *Then, there exists a* $\tau_0 > 0$ *such that the estimate from Lemma 3.3 also holds true for* $\tau < \tau_0$.

Proof. The estimate follows from taking τ sufficiently small (note that $\tau^{-\alpha}\tau \to 0$ as $\tau \to 0$) and the discrete Grönwall lemma [13, Corollary 15.5] after considering that

$$\left|\sum_{i=1}^{j} \langle \delta f_i, u_{i-1}\rangle_{\mathrm{H}_0^1(\Omega)^*\times \mathrm{H}_0^1(\Omega)}\tau\right| \leqslant C\left(1 + \sum_{i=1}^{j} t_i^{-\alpha}\,\|u_{i-1}\|^2_{\mathrm{H}_0^1(\Omega)}\,\tau\right) \leqslant C\left(1 + \sum_{i=1}^{j} t_i^{-\alpha}\,\|\nabla u_i\|^2\,\tau + \tau^{-\alpha}\tau \sum_{i=1}^{j} \|\nabla u_i - \nabla u_{i-1}\|^2\right).$$

$\square$

Now, we define the functions $z_n, \overline{z}_n : [0,T] \to \mathrm{L}^2(\Omega)$ by

$$z_n(t) = \begin{cases} z_0 & t = 0 \\ z_{i-1} + (t - t_{i-1})\delta z_i & t \in (t_{i-1}, t_i], \quad 1 \leqslant i \leqslant n, \end{cases} \qquad \text{and} \qquad \overline{z}_n(t) = \begin{cases} z_0 & t = 0 \\ z_i & t \in (t_{i-1}, t_i], \quad 1 \leqslant i \leqslant n. \end{cases}$$

Using these so-called Rothe's functions and Eq. (9), Eq. (7) can be rewritten on the whole time frame as

$$\langle \partial_t (k * u)_n(t) - \overline{k}_n(t)u_0, \varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times \mathrm{H}_0^1(\Omega)} + \overline{\mathcal{L}}_n(t)(\overline{u}_n(t), \varphi) = \left(\overline{f}_n(t), \varphi\right), \quad \forall \varphi \in \mathrm{H}_0^1(\Omega), \tag{10}$$

where

$$\overline{\mathcal{L}}_n(t)\,(\overline{u}_n(t), \varphi) = \left(\overline{\boldsymbol{A}}_n(t)\nabla\overline{u}_n(t) + \overline{\boldsymbol{b}}_n(t)\overline{u}_n(t), \nabla\varphi\right) + (\overline{c}_n(t)\overline{u}_n(t), \varphi)\,.$$

Theorem 3.1 (Existence). *Suppose that the conditions of Lemma 3.3 or 3.4 are fulfilled. Then there exists a unique weak solution u to the problem* (5) *with* $u \in \mathrm{C}\left([0,T], \mathrm{H}_0^1(\Omega)^*\right) \cap \mathrm{L}^\infty\left((0,T), \mathrm{H}_0^1(\Omega)\right)$ *and* $k * \partial_t u \in \mathrm{L}^2\left((0,T), \mathrm{H}_0^1(\Omega)^*\right)$.

Proof. From the Rellich-Kondrachov theorem [14, Theorem 6.6-3], we have that $\mathrm{H}_0^1(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}^2(\Omega)$. Lemma 3.2 gives that $(\overline{u}_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^2\left((0,T), \mathrm{H}_0^1(\Omega)\right)$. Thus we have the existence of an element $u \in \mathrm{L}^2\left((0,T), \mathrm{L}^2(\Omega)\right)$ and a subsequence $(\overline{u}_{n_l})_{l\in\mathbb{N}}$ of $(\overline{u}_n)_{n\in\mathbb{N}}$ such that

$$\overline{u}_{n_l} \to u \text{ in } \mathrm{L}^2\left((0,T), \mathrm{L}^2(\Omega)\right) \text{ as } l \to \infty.$$

From Lemma 3.2 and the reflexivity of the space $\mathrm{L}^2\left((0,T), \mathrm{H}_0^1(\Omega)\right)$, we have the existence of a subsequence (indexed by n_l again) such that

$$\overline{u}_{n_l} \rightharpoonup u \text{ in } \mathrm{L}^2\left((0,T), \mathrm{H}_0^1(\Omega)\right) \text{ as } l \to \infty. \tag{11}$$

Now, we integrate Eq. (10) in time over $(0,\eta) \subset (0,T)$ for the resulting subsequence to get that

$$\langle (k*u)_{n_l}(\eta), \varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times\mathrm{H}_0^1(\Omega)} - \int_0^\eta \langle u_0 \overline{k}_{n_l}(t), \varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times\mathrm{H}_0^1(\Omega)}\,\mathrm{d}t + \int_0^\eta \overline{\mathcal{L}}_{n_l}(t)(\overline{u}_{n_l}(t),\varphi)\,\mathrm{d}t = \int_0^\eta \langle \overline{f}_{n_l}(t),\varphi\rangle\,\mathrm{d}t. \tag{12}$$

We only point out the limit transition in the first term by showing that

$$\lim_{l\to\infty}\left|\int_0^T \langle (k*u)_{n_l}(t),\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times\mathrm{H}_0^1(\Omega)}\,\mathrm{d}t - \int_0^T \langle (k*\overline{u}_{n_l})(t),\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times\mathrm{H}_0^1(\Omega)}\,\mathrm{d}t\right| = 0. \tag{13}$$

This limit transition can be done in two steps

$$\text{(i)} \lim_{l\to\infty}\left|\int_0^T \langle (k*u)_{n_l}(t)\,\mathrm{d}t - \int_0^T \overline{(k*u)}_{n_l}(t),\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times\mathrm{H}_0^1(\Omega)}\,\mathrm{d}t\right| = 0;$$

$$\text{(ii)} \lim_{l\to\infty}\left|\int_0^T \langle\overline{(k*u)}_{n_l}(t)\,\mathrm{d}t - \int_0^T (k*\overline{u}_{n_l})(t),\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times\mathrm{H}_0^1(\Omega)}\,\mathrm{d}t\right| = 0.$$

The limit transition (i) follows from Lemma 3.2 and the Lebesgue dominated theorem as follows

$$\begin{aligned}\int_0^T \left|\langle (k*u)_{n_l}(t) - \overline{(k*u)}_{n_l}(t),\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times\mathrm{H}_0^1(\Omega)}\right|\mathrm{d}t &= \sum_{i=1}^{n_l}\int_{t_{i-1}}^{t_i}\left|\langle (t-t_i)\delta(k*u)_i,\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times\mathrm{H}_0^1(\Omega)}\right|\mathrm{d}t\\ &\overset{(9)}{\leqslant} \sum_{i=1}^{n_l}\tau_{n_l}^2\left|\langle (k*\delta u)_i - k_i u_0,\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times\mathrm{H}_0^1(\Omega)}\right| \leqslant C\tau_{n_l} + C\int_0^1 \tau_{n_l}^{1-\beta}\mu(\beta)\,\mathrm{d}\beta \overset{l\to\infty}{\longrightarrow} 0.\end{aligned}$$

For the limit transition (ii), we deduce that ($\lceil t\rceil_\tau = t_i$ when $t \in (t_{i-1}, t_i]$)

$$\begin{aligned}\int_0^T &\left|\langle\overline{(k*u)}_{n_l}(t) - (k*\overline{u}_{n_l})(t),\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times\mathrm{H}_0^1(\Omega)}\right|\mathrm{d}t = \int_0^T\left|\langle(\overline{k}_{n_l}*\overline{u}_{n_l})(\lceil t\rceil_\tau) - (k*\overline{u}_{n_l})(t),\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times\mathrm{H}_0^1(\Omega)}\right|\mathrm{d}t\\ &\leqslant \int_0^T\left(\left|\langle\int_0^t\left(\overline{k}_{n_l}(\lceil t\rceil_\tau - s) - k(t-s)\right)\overline{u}_{n_l}(s)\ \mathrm{d}s,\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times\mathrm{H}_0^1(\Omega)}\right| + \left|\langle\int_t^{\lceil t\rceil_\tau}\overline{k}_{n_l}(\lceil t\rceil_\tau - s)\,\overline{u}_{n_l}(s)\ \mathrm{d}s,\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times\mathrm{H}_0^1(\Omega)}\right|\right)\mathrm{d}t \to 0\end{aligned}$$

as $l \to \infty$ by applying Hölder's inequality twice on both terms on the right-hand side, since from Lemma 3.2, it follows that (for $t \in (t_{i-1}, t_i]$)

$$\int_t^{\lceil t\rceil_\tau}\overline{k}_{n_l}(\lceil t\rceil_\tau - s)\left\|\overline{u}_{n_l}(s)\right\|^2_{\mathrm{H}_0^1(\Omega)^*}\ \mathrm{d}s \leqslant C\int_0^{t_i}\overline{k}_{n_l}(t_i - s)\left\|\overline{u}_{n_l}(s)\right\|^2\ \mathrm{d}s = C\left(k*\|u\|^2\right)_i \leqslant C,$$

and

$$\int_0^T \left(\int_0^t \left| \overline{k}_{n_l}(\lceil t \rceil_\tau - s) - k(t-s) \right| \left\| \overline{u}_{n_l}(s) \right\|^2_{\mathrm{H}^1_0(\Omega)^*} \, \mathrm{d}s \right) \mathrm{d}t$$

$$\overset{(\star)}{\leqslant} \|k\|_{\mathrm{L}^1(0,T)} \left\| \overline{u}_{n_l} \right\|_{\mathrm{L}^2\left((0,T),\mathrm{H}^1_0(\Omega)^*\right)} + \int_0^T \left(\int_0^t \overline{k}_{n_l}(\lceil t \rceil_\tau - s) \left\| \overline{u}_{n_l}(s) \right\|^2_{\mathrm{H}^1_0(\Omega)^*} \, \mathrm{d}s \right) \mathrm{d}t \leqslant C,$$

where we used Young's inequality for convolutions at position $(\star)$; and since for $t \in (t_{i-1}, t_i]$, by the Lebesgue dominated theorem, it holds that

$$\int_t^{\lceil t \rceil_\tau} \overline{k}_{n_l}(\lceil t \rceil_\tau - s) \, \mathrm{d}s = \int_0^{t_i - t} \left(\int_0^1 \frac{\tau_{n_l}^{-\beta}}{\Gamma(1-\beta)} \mu(\beta) \, \mathrm{d}\beta \right) \mathrm{d}\xi \leqslant \int_0^1 \tau_{n_l}^{1-\beta} \mu(\beta) \, \mathrm{d}\beta \to 0 \quad \text{ as } \tau_{n_l} \to 0,$$

and

$$\int_0^t \left| \overline{k}_{n_l}(\lceil t \rceil_\tau - s) - k(t-s) \right| \, \mathrm{d}s \leqslant C \int_0^1 \tau_{n_l}^{1-\beta} \mu(\beta) \, \mathrm{d}\beta \to 0 \quad \text{ as } \tau_{n_l} \to 0.$$

Now, we integrate the equation (12) again in time over $\eta \in (0, \xi) \subset (0, T)$. Then, using Eq. (11) and (13), we are allowed to pass to the limit for $l \to \infty$. We get that ($\overline{k}_{n_l} \to k$ pointwise in $(0,T)$)

$$\int_0^\xi \langle (k * u)(\eta), \varphi \rangle_{\mathrm{H}^1_0(\Omega)^* \times \mathrm{H}^1_0(\Omega)} \, \mathrm{d}\eta - \int_0^\xi \int_0^\eta \langle u_0 k(t), \varphi \rangle_{\mathrm{H}^1_0(\Omega)^* \times \mathrm{H}^1_0(\Omega)} \, \mathrm{d}t \, \mathrm{d}\eta$$

$$+ \int_0^\xi \int_0^\eta \mathcal{L}(t)(u(t), \varphi) \, \mathrm{d}t \, \mathrm{d}\eta = \int_0^\xi \int_0^\eta (f(t), \varphi) \, \mathrm{d}t \, \mathrm{d}\eta.$$

Differentiating this relation with respect to ξ leads to

$$\langle (k * u)(\xi), \varphi \rangle_{\mathrm{H}^1_0(\Omega)^* \times \mathrm{H}^1_0(\Omega)} - \int_0^\xi \langle u_0 k(t), \varphi \rangle_{\mathrm{H}^1_0(\Omega)^* \times \mathrm{H}^1_0(\Omega)} \, \mathrm{d}t + \int_0^\xi \mathcal{L}(t)(u(t), \varphi) \, \mathrm{d}t = \int_0^\xi (f(t), \varphi) \, \mathrm{d}t. \tag{14}$$

i.e. $(k * u)(t) \in \mathrm{H}^1_0(\Omega)^*$ for all $t \in (0,T)$. The reasoning up to now is valid when $u_0 \in \mathrm{L}^2(\Omega)$. From Lemma 3.3 or 3.4, we obtain that $u \in \mathrm{L}^\infty\left((0,T), \mathrm{H}^1_0(\Omega)\right)$. Then, from Eq. (14), we have that

$$\lim_{\xi \to 0^+} \langle (k * u)(\xi), \varphi \rangle_{\mathrm{H}^1_0(\Omega)^* \times \mathrm{H}^1_0(\Omega)} = 0 \quad \Rightarrow \quad (k * u)(0) = 0 \text{ in } \mathrm{H}^1_0(\Omega)^*.$$

Moreover, differentiating (14) with respect to ξ (and replacing ξ by t) gives

$$\langle \partial_t (k * u)(t) - k(t) u_0, \varphi \rangle_{\mathrm{H}^1_0(\Omega)^* \times \mathrm{H}^1_0(\Omega)} + \mathcal{L}(t)\,(u(t), \varphi) = \langle f(t), \varphi \rangle_{\mathrm{H}^1_0(\Omega)^* \times \mathrm{H}^1_0(\Omega)}, \qquad \forall \varphi \in \mathrm{H}^1_0(\Omega). \tag{15}$$

Thus u is satisfying (5) since $\partial_t(k * u) - k u_0 = k * \partial_t u \in \mathrm{L}^2\left((0,T), \mathrm{H}^1_0(\Omega)^*\right)$ as u turns out to be absolutely continuous. Since $\partial_t (k * u)(t) - k(t) u_0 = \partial_t \left(k * (u - u_0)\right)(t)$ and $\left(k * (u - u_0)\right)(0) = 0$ in $\mathrm{H}^1_0(\Omega)^*$, integrating (15) in time gives that $(k * (u - u_0))(t)$ is absolutely continuous with values in $\mathrm{H}^1_0(\Omega)^*$. Following [10, Proposition 6], there exists a non-negative kernel $g \in \mathrm{L}^1(0,T)$ such that $g * k = 1$ and thus

$$(g * \partial_t \left(k * (u - u_0)\right))(t) = \partial_t \left(g * k * (u - u_0)\right)(t) = u(t) - u(0) \quad \text{ in } \mathrm{H}^1_0(\Omega)^*.$$

Therefore, applying this convolution operation on (15) gives that u is absolutely continuous in the time variable, i.e. $u \in \mathrm{C}\left([0,T], \mathrm{H}^1_0(\Omega)^*\right)$. We conclude the proof by showing the uniqueness of a solution. We put $\varphi = u(t)$ in (5) with $u_0 = f = 0$ and integrate over the time interval $(0,T)$. We obtain that

$$\int_0^T \langle \partial_t \left(k * u\right)(t), u(t) \rangle_{\mathrm{H}^1_0(\Omega)^* \times \mathrm{H}^1_0(\Omega)} \, \mathrm{d}t + \int_0^T \mathcal{L}(t)\,(u(t), u(t)) \, \mathrm{d}t = 0.$$

From [10, Corollary 16], it follows that

$$\frac{k(T)}{2} \int_0^T \|u(t)\|^2 \, \mathrm{d}t + \frac{\alpha}{2} \int_0^T \|\nabla u(t)\|^2 \, \mathrm{d}t = 0,$$

i.e. $u = 0$ a.e. in Q_T. □

Remark 3.1. *The result of Theorem 3.1 is also valid for the fractional diffusion equation (i.e. when* $k(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$*), which gives an improvement of the result obtained in [15, Theorem 2 and 4].*

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